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AUTOMATIC DAMPING OF VIBRATIONS OF AN AIRPLANE WING BY INTERNAL CONTROL FORCES

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An increase in the absolute dimensions of aircraft leads to a decrease in their dynamic rigidity. Both the frequency of natural vibrations and the structural damping factor are decreased. The deformations produced by impulsive forces die down slowly, but periodic disturbances may increase as a result of resonance. All this leads to a decrease of the flying life of the structure. We study various methods of damping elastic vibrations by using internal forces. The amplitude, frequencies, and phase of the forces acting are governed by a control system. A movable mass, an internal tension, a flexible shaft, and a gyromotor are considered as a control. In contrast with the familiar method using external aerodynamic forces, an internal control continues to be effective also on the airfield where, as it turns out, the airplane is subjected to the largest dynamic load.

1. Flexural-torsional vibrations of a wing of small sweepback and large aspect ratio can be described by a two-component vector function $\{w(y, t), \theta(y, t)\}$ [1]. Here $w(y, t)$ is the vertical displacement from the equilibrium position of the elastic axis of the wing, and $\theta(y, t)$ is the rotation of a chord of the wing about the elastic axis. These quantities are functions of the coordinate y of a cross section and the time t. In terms of these variables,

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free flexural—torsional vibrations of a wing are described by the following system of ordinary differential equations:

$$m\ddot{w} - s_y\ddot{\theta} + (EIw'')'' = 0; \tag{1.1}$$

$$J_y\ddot{\theta} - s_y\ddot{w} - (GJ\theta')' = 0, \tag{1.2}$$

where $m(y, t)$ is the distribution of the mass of the wing along the axis, including the mass of the engine nacelles and the fuel tanks; burnup of fuel makes this mass a slowly varying function of time. If a movable mass in the wing is used for control, a rapidly varying term appears.

$J_y(y, t)$ is the polar moment of intertia of the wing with respect to the elastic axis; s_y is the static moment, determined by the product of the mass m times the distance between the elastic axis and the axis of inertia.

The products EI and GJ represent the flexural and torsional rigidities of the wing.

In Eqs. (1.1) dots over letters denote time derivatives, and primes mean derivatives with respect to the spatial variable y.

The end of the wing fastened to the fuselage satisfies the boundary conditions

$$w = w' = \theta = 0 \quad \text{at} \quad y = 0.$$

The free end of the wing satisfies the following conditions:

$$\left.\begin{aligned} EIw'' = (EIw'') = 0 \quad (1.3)\\ GJ\theta' = 0 \quad (1.4)\end{aligned}\right\} \quad \text{at} \quad y = L$$

where L is the length of the wing.

It is known that the vector function of free vibrations can be sought in the form

$$e^{i\omega_0 t}\{f_w(y),\ f_\theta(y)\},$$

where ω_0 is one of the natural frequencies, and f_w and f_θ are the eigenfunctions of the vibrations of the elastic structure.

A natural frequency ω_0 satisfies the relation

$$\omega_0^2 = \frac{\int\limits_0^L \left[EI\left(f_w''\right)^2 + GJ\left(f_\theta'\right)^2\right] dy}{\int\limits_0^L \left(mf_w^2 - 2s_y f_w f_\theta + J_y f_\theta^2\right) dy}, \tag{1.5}$$

which we shall use later. We consider only one frequency and one vibrational mode, but since the problem is linear, the superposition principle can be used to extend the results obtained to the general case.

By considering only small control actions it can be assumed that the shape of the deformation is described by the same eigenfunctions $\{f_w(y),\ f_\theta(y)\}$ as for free vibrations, and that only the natural frequency ω_0 is changed by the addition of a negative imaginary part. The change in the real part of ω_0 is of second order. We shall refine the concept of a small quantity later.

2. A moving mass in a wing is most simply realized by the rotation of a concentrated mass m_0 in a circle about a certain point $y = l_1$. If the radius of the circle along which the mass moves is r, and the frequency of rotation is ω_1, the distributed mass of the wing m is increased by a function of the time and coordinate $m_1 = m_0\delta[y - l_1 + r \sin \omega_1(t - t_0)]$, where δ is the delta function, which becomes infinite at $y = l_1 - r \sin \omega_1(t - t_0)$. In addition, the moment of inertia of the wing is increased by

$$J_1 = m_0 r^2 \sin^2\omega_1(t - t_0)\delta[y - l_1 + r \sin \omega_1(t - t_0)].$$

For this case the equations for flexural—torsional vibrations of the wing take the form

$$\begin{aligned} m\ddot{w} - s_y\ddot{\theta} + d/dt(m_1\dot{w}) - d/dt(sm_1\dot{\theta}) + (EIw'')'' = 0,\\ J_y\ddot{\theta} - s_y\ddot{w} + d/dt(J_1\dot{\theta}) - d/dt(sm_1\dot{w}) - (GJ\theta')' = 0,\end{aligned} \tag{2.1}$$

where s is the distance between the elastic axis and the axis of inertia of the wing, which we assume is fixed.

Since the control actions are small, the solution of system (2.1) with boundary conditions (1.3) and (1.4) can be sought in the form

$$w(y,\ t) = A(t)f_w(y), \quad \theta = A(t)f_\theta(y),$$

where $\{f_w(y),\ f_\theta(y)\}$ is the vector function of free vibrations, and A(t) is an unknown function of the time. This function can be determined by using the energy equation. To do this, we multiply the first of Eqs. (2.1) by w, the second by θ, add, and integrate with respect to y from 0 to L. By using boundary conditions (1.3), (1.4), Eq. (1.5), and the expressions for w and θ, we obtain an ordinary differential equation for A

$$\frac{d^2A}{dt^2} + \omega_0^2 A + \frac{\frac{d}{dt}\left[\frac{dA}{dt}\int\limits_0^L \left(m_1 f_w^2 + J_1 f_\theta^2 - 2sm_1 f_w f_\theta\right) dy\right]}{\int\limits_0^L \left(m f_w^2 + J_y f_\theta^2 - 2s_y f_\theta f_w\right) dy}. \tag{2.2}$$

To within second-order terms in m_1 and J_1 the denominator can be evaluated for constant functions m(y) and $J_y(y)$ without taking account of the variable part introduced by the moving mass

$$N = \int\limits_0^L \left(m f_w^2 + J_y f_\theta^2 - 2s_y f_\theta f_w\right) dy.$$

The numerator can be considerably simplified. To do this we first evaluate the integral appearing in it by using the fact that the functions m_1 and J_1 are delta functions

$$Q = \int\limits_0^L \left(m_1 f_w^2 + J_1 f_\theta^2 - 2sm_1 f_\theta f_w\right) dy = m_0\left[f_w^2\left(l_1 - r\sin\omega_1 t\right)\right.$$
$$\left. + r^2\sin^2\omega_1 t f_\theta^2\left(l_1 - r\sin\omega_1 t\right) - 2sf_w\left(l_1 - r\sin\omega_1 t\right) f_\theta\left(l_1 - r\sin\omega_1 t\right)\right.$$

Since the radius r is appreciably smaller than the distance between nodes of the eigenfunction $\{f_w,\ f_\theta\}$, we can write

$$f_w\left(l_1 - r\sin\omega_1 t\right) = f_w\left(l_1\right) - r\sin\omega_1 t f_w'\left(l_1\right),$$
$$f_\theta\left(l_1 - r\sin\omega_1 t\right) = f_\theta\left(l_1\right) - r\sin\omega_1 t f_\theta'\left(l_1\right).$$

Neglecting second-order quantities, we obtain $Q = M_1 - M_2 \sin \omega_1 t$, where

$$M_1 = m_0\left[f_w^2\left(l_1\right) - 2sf_\theta\left(l_1\right) f_w\left(l_1\right)\right];$$
$$M_2 = m_0\left[2rf_w\left(l_1\right) f_w'\left(l_1\right) - 2srf_w' f_\theta - 2srf_\theta'\left(l_1\right) f_w\left(l_1\right)\right].$$

Thus, Eq. (2.2) takes the form

$$\frac{d^2A}{dt^2} + \Omega^2 A - \frac{M_2\omega_1}{N}\cos\omega_1 t\,\frac{dA}{dt} = 0, \tag{2.3}$$

where

$$\Omega^2 = \omega_0^2\left(1 - \frac{M_1}{N\omega_0^2}\right).$$

We note that the control action entered the coefficient in Eq. (2.3). Such control is called parametric.

Equation (2.3) can be solved by the van der Pol method [2].

To do this we set x = A, z = dA/dt, and rewrite Eq. (2.3) in the form

$$dx/dt = z,$$
$$dz/dt + \Omega^2 x - \left(M_2\omega_1 z/N\right)\cos\omega_1 t = 0, \tag{2.4}$$

whose solution we seek in the form

$$x = a(t)\cos\left[\Omega t + \chi(t)\right],\ z = -a(t)\omega_1\sin\left[\Omega t + \chi(t)\right], \tag{2.5}$$

where $a(t)$ and $\chi(t)$ are unknown solely varying functions of the time.

After substituting (2.5) into (2.4), we obtain

$$\frac{da}{dt}\cos\psi - a\frac{d\chi}{dt}\sin\psi = 0,$$
$$\frac{da}{dt}\sin\psi + a\frac{d\chi}{dt}\cos\psi = \frac{aM_2\omega_1}{N}\cos\omega_1 t\sin\psi, \tag{2.6}$$

where $\psi = \Omega t + \chi(t)$. Elimination of $d\chi/dt$ from the last system gives

$$\frac{da}{dt} = \frac{aM_2\omega_1}{N}\cos\omega_1 t\sin^2\psi. \tag{2.7}$$

According to van der Pol's method, the coefficient in Eq. (2.7) can be averaged over time, or what amounts to the same thing, over the variable ψ. The result of the averaging depends on the frequency and phase of the control, which so far remain arbitrary.

It is easy to see that the average value will be different from zero only if the control frequency is 2Ω.

Choosing the phase of the control to coincide with that of χ, we obtain $da/dt = -aM_2\Omega/2N$.

Hence it follows that the amplitude of the vibrations will decrease according to the law

$$a(t) = a(0)\exp[-\alpha\Omega t],$$

where

$$\alpha = \frac{m_0 r\left[f_w(l_1)f_w'(l_1) - sf_w'(l_1)f_\theta(l_1)\right]}{2\int\limits_0^L\left(mf_w^2 + J_y f_\theta^2 - 2s_y f_w f_\theta\right)dy}.$$

To find the phase of χ we first solve (2.6) for $d\chi/dt$,

$$d\chi/dt = (M_2\Omega/N)\cos\omega_1 t\sin 2\psi.$$

It is clear from this that the phase of χ is a slowly varying function which remains constant on the average.

3. Let us solve the problem of the damping of vibrations of a wing by producing an internal longitudinal stress. In this case the equations for flexural–torsional vibrations of the wing can be written in the form

$$m\ddot{w} - s_y\ddot{\theta} - Tw'' + (EIw'')'' = 0, \quad J_y\ddot{\theta} - s_y\ddot{w} - (GJ\theta')' = 0,$$

where $T(t)$ is the cable tension.

As before we use the energy method to reduce the problem to that of solving an ordinary differential equation for $A(t)$

$$\frac{d^2A}{dt^2} + \omega_0^2(1-\tau)A(t) = 0,$$

where

$$\tau = \frac{T\left\{f_w'(L)f(L) - \int\limits_0^L[f'(y)]^2dy\right\}}{\omega_0^2\int\limits_0^L\left(mf_w^2 - 2s_y f_w f_\theta + J_y f_\theta^2\right)dy}.$$

We write $A(t)$ in the form $A(t) = a(t)\cos(\omega_0 t - \chi)$, where $a(t)$ is the slowly varying amplitude of the vibrations. Using van der Pol's method we find the equation for the amplitude $a(t)$

$$\frac{da}{dt} + \frac{\tau(t)\omega_0 a}{2}\sin 2\psi = 0.$$

It is clear from the equation that the control law $\tau(t)$ must be chosen in the form

$$\tau(t) = \tau_0\sin 2\psi.$$

Then the equation for the amplitude with a time averaged coefficient takes the form

$$da/dt + {}^1/_4\tau_0\omega_0 a = 0.$$

Hence $a(t) = a(0)\exp[-{}^1/_4\tau_0\omega_0 t]$.

In this case the damping factor is given by

$$\alpha = \frac{1}{4}\tau_0 = \frac{T\left\{f_w'(L)\,f_w(L) - \int\limits_0^L \left[f_w'(y)\right]^2 dy\right\}}{4\omega_0^2\int\limits_0^L \left(mf_w^2 - 2s_y f_w f_\theta + J_y f_\theta^2\right) dy}.$$

By using Eq. (1.5) the denominator can be expressed in terms of the potential energy of the deformed wing. In addition, the integral of the square of a small quantity can be neglected in the numerator. Then

$$\alpha = \frac{Tf_w'(L)\,f_w(L)}{4\int\limits_0^L \left[EI\,(f_w'')^2 + GJ\,(f_\theta')^2\right] dy}.$$

The products Tf_w' represents the projection of the force T on the transverse coordinate. In this case α turns out to be equal to one-fourth the ratio of the work done by the control force to the potential energy of the deformed wing.

The requirement $\alpha \ll 1$, on the one hand, is necessary for the validity of the approximations made above, and on the other hand, permits the treatment of small control forces. At the same time, an extremely small α will correspond to slowly damped vibrations.

It is noted that the mechanisms considered for damping vibrations retain their effectiveness even when the flexure of the wing is not accompanied by torsion. In this case the damping factor for a movable mass will have the value

$$\alpha = \frac{m_0 r f_w(l_1)\,f_w'(l_1)}{N}.$$

For control by using a variable internal stress

$$\alpha = \frac{Tf_w'(L)\,f_w(L)}{\omega^2\int\limits_0^L mf_w^2 dy}.$$

4. We now consider damping by a moment produced by the bending of a flexible shaft passing through the whole wing. One end of the shaft is connected to the outer end of the wing, and the other end, located in the airplane fuselage, can be twisted according to a definite law by a hydraulic device.

The vibration of a wing acted on by a moment applied at the cross section $y = l$ can be written in the form

$$m\ddot{w} - s_y\ddot{\theta} + (EIw'')'' = 0,$$

$$J_y\ddot{\theta} - s_y\ddot{w} - (GJ\theta')' = M\delta(l - y),$$

where M(t) is the moment, and $\delta(l - y)$ is the delta function.

Starting from the assumption that the control action is small, we seek the solution in the usual form. For the function A(t) we obtain the equation

$$\frac{d^2A}{dt^2} - \omega_0^2 A = \frac{M(t)\,f_\theta(l)}{\int\limits_0^L \left(mf_w^2 + J_y f_\theta^2 - 2s_y f_\theta f_w\right) dy}.$$

We determine the control moment from the law

$$M = -M_0\omega_0 f_\theta(l)\,dA/dt.$$

In this case the function A(t) will decrease according to the law

$$A(t) = A(0)\exp[-\alpha\omega_0 t],$$

where

$$\alpha = M_0 f_\theta^2/N.$$

The maximum value of the torque M_0 is determined by the product of the rigidity of the shaft times the torsion angle, and of course by the tensile strength. The possibility of twisting the shaft through a large angle limited only by the tensile strength permits maximum use of the shaft material and thus a decrease in its weight.

Control by a gyroscopic moment is of particular interest. While all the preceding cases required a system of vibration transducers, amplifiers, and force mechanisms, a flywheel, or better stated, a gyromotor, permits the combination of all the functions of the control system to achieve maximum simplicity and reliability.

We denote by K the angular momentum of the flywheel. For a large angular momentum and a small angular velocity of precession of its axis, it is possible to use the law of motion of the axis [3]

$$K\omega_y = M_x, \quad K\omega_x = -M_y,$$

where ω_x and ω_y are the angular velocities of the axis of the flywheel about the x and y axes, and M_x and M_y are the components of the applied moment.

In order to damp flexural—torsional vibrations of a wing, the flywheel must be coupled to the wing in such a way that the torsional vibrations of the wing excite the same rotations of the axis of the flywheel about the y axis, i.e.,

$$\omega_y = \dot{\theta}(l_1, t) = A(t) f_\theta(l_1).$$

We denote by l_1 the y coordinate of the position of the flywheel.

We shall specify the angular velocity ω_x in a special way depending on the purpose and means of control. We locate the undisturbed position of the axis of the flywheel perpendicular to the elastic axis of the wing.

We write the component of the angular velocity ω_x as the sum

$$\omega_x = \partial^2 w/\partial x \partial t + \dot{\varphi}(t).$$

Here the mixed derivative is the angular velocity of flexure of the wing axis, and $\dot{\varphi}$ is the angular velocity of rotation of the axis of the flywheel relative to the bent wing. While the first term is not at our disposal, the velocity $\dot{\varphi}(t)$ can be chosen at our discretion depending on the purposes and means of control.

A sufficiently general control law can be specified by the expression

$$\dot{\varphi} = p\dot{\theta} + \omega_0 q\theta, \tag{4.1}$$

where p and q are certain dimensionless coefficients.

The equations of vibrations of a wing acted upon by gyroscopic moments can be written in the form

$$m\ddot{w} - s_y\ddot{\theta} - [M_x\delta(l_1 - y)]' + (EIw'')'' = 0,$$
$$J_y\ddot{\theta} - s_y\ddot{w} - M_y\delta(l_1 - y) - (GJ\theta')' = 0.$$

The energy equation leads to the following equation for the amplitude A(t):

$$\frac{d^2A}{dt^2} + \omega_0^2 A = \frac{M_x f_w'(l_1) + M_y f_\theta(l_1)}{\int\limits_0^L \left(m f_w^2 - 2s_y f_w f_\theta + J_y f_\theta^2\right) dy}.$$

We substitute into this the values of the moments M_x and M_y

$$M_x = K\omega_y = K f_\theta(l_1)\frac{dA}{dt}, \quad M_y = -K\omega_x = -K\frac{\partial^2 w}{\partial t \partial x} - K\dot{\varphi} = K f_w'(l_1)\frac{dA}{dt} - Kp f_\theta(l_1)\frac{dA}{dt} - KqA f_\theta(l_1)\omega_0.$$

As a result, we obtain

$$\frac{d^2A}{dt^2} + \omega_0\alpha\frac{dA}{dt} + \Omega^2 A = 0, \tag{4.2}$$

where

$$\Omega^2 = \omega_0^2\left[1 + \frac{Kqf_\theta^2(l_1)}{\omega_\theta^2\int\limits_0^L\left(mf_w^2 + J_yf_\theta^2 - 2s_yf_\theta f_w\right)dy}\right];$$

$$\alpha = \frac{K\left[f_w'(l_1) + pf_\theta(l_1)\right]f_\theta(l_1)}{\omega_0\int\limits_0^L\left(mf_w^2 + J_yf_\theta^2 - 2s_yf_\theta f_w\right)dy}.$$

The following expressions for Ω and α are sufficiently accurate:

$$\Omega = \omega_0, \quad \alpha = \frac{Kpf_\theta^2(l_1)}{\omega_0\int\limits_0^L\left(mf_w^2 + J_yf_\theta^2 - 2s_yf_wf_\theta\right)dy}. \tag{4.3}$$

Let us consider the question of the physical realization of the control law (4.1) chosen.

The presence of the term $\omega_0 q\theta$ in (4.1) leads to a certain change in the frequency of vibrations, and its realization requires an actuator with feedback. As always, the damping is due to a term proportional to the strain rate, in the present case $p\dot{\theta}$.

Rotations about the y axis connected with the wing give rise to a gyroscopic moment about the Ox axis. If now the flywheel housing is connected to the wing through a damper which permits rotation about the x axis according to the law $M_x = \varkappa\dot{\varphi}$, we ensure the required relation

$$K\dot{\theta} = \varkappa\dot{\varphi}. \tag{4.4}$$

Hence the factor p in Eqs. (4.1)-(4.3) will be given by

$$p = K/\varkappa.$$

It is desirable that p be as large as possible, but it is restricted by kinematic conditions. Let us set $\theta = \theta_0 \exp(i\omega t)$, $\varphi = \varphi_0 \exp(i\omega t)$. Then Eq. (4.4) gives a relation for the amplitudes of the vibrations

$$K\theta_0 = \varkappa\varphi_0.$$

The amplitude of the rotation of the axis of the flywheel cannot exceed 90°, since in this position the flywheel could not produce the required gyroscopic moment. The formulas used generally assume that the angle $\varphi_0 < 30°$. The amplitude θ_0 is determined by operating conditions. Hence it is necessary to choose the damping factor $\varkappa$, so that the ratio $K/\varkappa < \varphi_0/\theta_0$, otherwise the flywheel will bump against the mechanical stops in the limiting positions. From the mathematical point of view this leads to a violation of the conditions of applicability of the description used, and from the technical point of view to decrease the effectiveness of the damping system.

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